

Heat-kernel coefficients and functional determinants for higher-spin fields on the ball

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August 1995

Abstract: *The zeta function associated with higher-spin fields on the Euclidean 4-ball is investigated. The leading coefficients of the corresponding heat-kernel expansion are given explicitly and the zeta functional determinant is calculated. For fermionic fields the determinant is shown to differ for local and spectral boundary conditions.*

PACS number(s): 02.30.-f, 04.62.+v

1 Introduction

Motivated by the need to give answers to some fundamental questions in quantum field theory, during the last years there has been and continues to be a lot of interest in the problem of calculating the heat-kernel coefficients and the determinant of a differential operator, L (see for example [1, 2, 3]). In mathematics the interest in the heat-kernel coefficients stems, in particular, from the well-known connections that exist between the heat-equation and the Atiyah-Singer index theorem [4]. The knowledge of the functional determinant allows one to obtain estimates of different types [5, 6, 7]. All these informations may be obtained by a knowledge of the zeta function $\zeta_L(s)$ associated with the operator L . For example, for the heat-kernel coefficients K_n associated with a positive, elliptic, second order differential operator L , one has the equation [8, 9]

$$K_n = \text{Res} \left[\zeta_L(s) \Gamma(s) \right] \Big|_{s=\frac{D-n}{2}} \quad (1.1)$$

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with the dimension D of the spacetime. The most appropriate way of dealing with the determinant of the operator L was introduced by Ray and Singer [10] and consists of defining

$$\ln \det L = -\zeta'_L(0). \quad (1.2)$$

Especially in recent times, there has been an increasing interest in these quantities with the 4-dimensional Euclidean ball as the underlying manifold and the relevant operator being the Laplacian respectively the operator describing higher-spin fields [11, 12, 13, 14, 15, 16] [17, 18, 19, 20].

The origin of this interest may be found in connection with quantum cosmology and supergravity [21, 22] (see also [23, 24] [25, 26, 27]). For example $\zeta(0)$ determines the scaling of the theory, $\zeta'(0)$ the one-loop effective action.

Very effective schemes have been developed for the calculation of these and similar quantities [11, 12, 13, 14, 15, 16, 20]. Up to now, the mentioned schemes have been applied mainly to the spin-0 case which means to the Laplacian on the Euclidean ball (see however [15]). The aim of this paper is to continue the consideration of [11, 12] and to apply the method developed there to higher-spin fields. For fermionic fields one has a choice between local and nonlocal spectral boundary conditions [28]. It was found that the $\zeta(0)$ values for both boundary conditions agree. Here we will see, that other properties of the zeta function as for example $\zeta'(0)$ and thus the effective action are different.

The outline of the paper is as follows. In section 2 we briefly describe the method developed in [11] for the calculation of the heat-kernel coefficients applying it to the spin- $\frac{1}{2}$ case with local boundary conditions. The value for $\zeta(0)$ found in [21] is confirmed. We describe how to obtain an arbitrary large number of the heat-kernel coefficients, summarizing the first ones in appendix A. The results for spectral boundary conditions and other spin fields are also summarized in appendix A. The way to obtain them is identical to the calculation shown for the spin- $\frac{1}{2}$ field with local boundary conditions and so only some small details are given at the beginning of the appendix A. In section 3 we direct our interest to the calculation of the functional determinant. Once more we choose as an example the spin- $\frac{1}{2}$ field with local boundary conditions to apply the formalism developed in [12]. The results for the other spins are then relatively easy obtained and are given without further considerations at the end of section 3. The results are seen to differ for local and spectral boundary conditions. In the conclusions we summarize the main results of our article.

2 Heat-kernel coefficients for higher-spin-fields on the 4-ball

To exemplify the method, we will choose the spin- $\frac{1}{2}$ field with local boundary conditions. Higher-spin fields can be treated in the same manner, with minor modifications (see the appendix A). As explained in detail in [21], the eigenvalues of the Dirac operator with local boundary conditions on the 4-dimensional ball are the solutions of the equation

$$J_{n+1}^2(E_{n,j}R) - J_{n+2}^2(E_{n,j}R) = 0, \quad n \geq 0, \quad (2.1)$$

where each eigenvalue $E_{n,j}$ carries degeneracy $(n+1)(n+2)$. Using Eq. (2.1) and following the lines of [11], the associated zeta function is written as a contour integral

$$\zeta_{1/2}^{lo}(s) = \sum_{n=0}^{\infty} (n+1)(n+2) \int_{\gamma} \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln[J_{n+1}^2(kR) - J_{n+2}^2(kR)], \quad (2.2)$$

where the contour γ may be chosen to run counterclockwise enclosing all solutions of Eq. (2.1) on the real positive axis. Here we introduced a mass parameter m , because it makes the analytical continuation procedure slightly easier. The limit $m \rightarrow 0$ will be taken at the end. As it stands, the representation (2.2) is valid for $\Re s > 2$. However, in order to determine the heat-kernel coefficients with higher index we need the properties of $\zeta_{1/2}^{lo}(s)$ in the range $\Re s < 0$ and thus we need to perform the analytical continuation to the left domain of the complex plane. Before considering in detail the n -summation, it is useful to first proceed with the k -integral alone.

The first specific idea is to shift the integration contour and place it along the imaginary axis. In order to avoid contributions coming from the origin $k = 0$, we will consider (with $\nu = n + 1$) the expression

$$\zeta_{1/2}^{lo,\nu}(s) = \int_{\gamma} \frac{dk}{2\pi i} (k^2 + m^2)^{-s} \frac{\partial}{\partial k} \ln \left(k^{-2\nu} [J_{\nu}^2(kR) - J_{\nu+1}^2(kR)] \right), \quad (2.3)$$

where the additional factor $k^{-2\nu}$ in the logarithm does not change the result, for no additional pole is enclosed. One then easily obtains

$$\zeta_{1/2}^{lo,\nu}(s) = \frac{\sin(\pi s)}{\pi} \int_m^{\infty} dk [k^2 - m^2]^{-s} \frac{\partial}{\partial k} \ln \left(k^{-2\nu} [I_{\nu}^2(kR) + I_{\nu+1}^2(kR)] \right) \quad (2.4)$$

valid in the strip $1/2 < \Re s < 1$.

As the next step of our method, we make use of the uniform expansion of the Bessel function $I_{\nu}(k)$ and its derivative for $\nu \rightarrow \infty$ as $z = k/\nu$ fixed [29]. It turns out, that things simplify if the Bessel functions in Eq. (2.4) are rewritten in combinations of Bessel functions with only one index. One may show using [30] (see also [21]) that

$$\begin{aligned} \zeta_{1/2}^{lo,\nu}(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln \left(z^{-2\nu} \left[I_{\nu}'^2(z\nu) \right. \right. \\ \left. \left. + \left(1 + \frac{1}{z^2} \right) I_{\nu}^2(z\nu) - \frac{2}{z} I_{\nu}(z\nu) I_{\nu}'(z\nu) \right] \right). \end{aligned} \quad (2.5)$$

Let us now employ the asymptotic expansion. One has [29]

$$I_{\nu}(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \Sigma_1 \quad (2.6)$$

with $t = 1/\sqrt{1+z^2}$ and $\eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$, furthermore

$$I_{\nu}'(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}(1+z^2)^{\frac{1}{4}}}{z} \Sigma_2, \quad (2.7)$$

where we introduced

$$\Sigma_1 = 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k}, \quad \Sigma_2 = 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k}. \quad (2.8)$$

The first few coefficients are listed in [29], higher coefficients are immediate to obtain by using the recursion relation given also there.

Actually we will need the asymptotics of (see Eq. (2.5))

$$\begin{aligned}
& \ln \left\{ I_\nu'^2(z\nu) + \left(1 + \frac{1}{z^2}\right) I_\nu^2(z\nu) - \frac{2}{z} I_\nu(z\nu) I'_\nu(\nu z) \right\} \\
& \sim \ln \left\{ \frac{(1+z^2)^{1/2} e^{2\nu\eta}}{2\pi\nu z^2} [\Sigma_1^2 + \Sigma_2^2 - 2t\Sigma_1\Sigma_2] \right\} \\
& = \ln \left\{ \frac{(1+z^2)^{1/2} e^{2\nu\eta}}{2\pi\nu z^2} 2(1-t) \right\} \\
& \quad + \ln \left\{ \frac{1}{2(1-t)} [\Sigma_1^2 + \Sigma_2^2 - 2t\Sigma_1\Sigma_2] \right\},
\end{aligned}$$

and for that reason we introduce the polynomials

$$\ln \left\{ \frac{1}{2(1-t)} [\Sigma_1^2 + \Sigma_2^2 - 2t\Sigma_1\Sigma_2] \right\} = \sum_{j=1}^{\infty} \frac{D_j(t)}{\nu^j}, \quad (2.9)$$

which are easily determined by a simple computer program.

Now comes the main idea of our approach. By adding and subtracting N leading terms of the asymptotic expansion, Eq. (2.9), for $\nu \rightarrow \infty$, Eq. (2.5) may be split into the following pieces

$$\zeta_{1/2}^{lo,\nu}(s) = Z_{1/2}^{lo,\nu}(s) + \sum_{i=-1}^N A_i^{1/2,\nu}(s), \quad (2.10)$$

with the definitions

$$\begin{aligned}
Z_{1/2}^{lo,\nu}(s) &= \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left\{ \ln \left[z^{-2\nu} \right. \right. \\
&\quad \times \left(I_\nu'^2(z\nu) + \left(1 + \frac{1}{z^2}\right) I_\nu^2(z\nu) - \frac{2}{z} I_\nu(z\nu) I'_\nu(\nu z) \right) \left. \right. \\
&\quad \left. \left. - \ln \left[\frac{(1+z^2)^{1/2} e^{2\nu\eta}}{2\pi\nu z^{2+2\nu}} \right] - \sum_{j=1}^N \frac{D_j(t)}{\nu^j} \right\} \right]
\end{aligned} \quad (2.11)$$

and

$$A_{-1}^{1/2,\nu}(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln \left(z^{-2\nu} e^{2\nu\eta} \right), \quad (2.12)$$

$$A_0^{1/2,\nu}(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \ln \frac{(1+z^2)^{1/2} (1-t)}{x^2}, \quad (2.13)$$

$$A_i^{1/2,\nu}(s) = \frac{\sin(\pi s)}{\pi} \int_{mR/\nu}^{\infty} dz \left[\left(\frac{z\nu}{R} \right)^2 - m^2 \right]^{-s} \frac{\partial}{\partial z} \left(\frac{D_i(t)}{\nu^i} \right). \quad (2.14)$$

The essential idea is conveyed here by the fact that the representation (2.10) has the following important properties. First, by considering the asymptotics of the integrand in Eq. (2.11) for

$z \rightarrow mR/\nu$ and $z \rightarrow \infty$, it can be seen that the function

$$Z_{1/2}^{lo}(s) = \sum_{\nu=1}^{\infty} \nu(\nu+1) Z_{1/2}^{lo,\nu}(s) \quad (2.15)$$

is analytic on the strip $(2-N)/2 < \Re s < 1$. For this reason, it gives no contribution to the residue of $\zeta_{1/2}^{lo}(s)$ in that strip. Furthermore, for $s = -k$, $k \in \mathbb{N}_0$, $k < -1 + N/2$, we have $Z_{1/2}^{lo}(s) = 0$ and, thus, it also yields no contribution to the values of the zeta function at these points. Together with Eq. (1.1) this result means that the heat-kernel coefficients are just determined by the terms $A_i^{1/2}(s)$ with

$$A_i^{1/2}(s) = \sum_{\nu=1}^{\infty} \nu(\nu+1) A_i^{1/2,\nu}(s). \quad (2.16)$$

As they stand, the $A_i^{1/2,\nu}(s)$ in eqs. (2.12), (2.13) and (2.14) are well defined on the strip $1/2 < \Re s < 1$ (at least). However, the way how to obtain the analytic continuation in the parameter s to the whole of the complex plane, in terms of known functions, has been explained recently [11] and will be exploited further here. Instead of repeating the analysis presented there, let us only mention that the analogy of the spin-1/2 field with the scalar field obeying Robin boundary conditions might be taken advantage off. The relevant numbers entering the final result are the coefficients in the polynomial $D_i(t)$,

$$D_i(t) = \sum_{a=0}^{2i} x_{i,a} t^{a+i}. \quad (2.17)$$

In terms of these and restricting to the massless case, we found

$$\begin{aligned} A_{-1}^{1/2}(s) &= \frac{R^{2s}}{2\sqrt{\pi}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s+1)} [\zeta_R(2s-2) + \zeta_R(2s-3)], \\ A_0^{1/2}(s) &= -\frac{R^{2s}}{2\sqrt{\pi}} \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma(s+1)} [\zeta_R(2s-1) + \zeta_R(2s-1)], \\ A_i^{1/2}(s) &= -\frac{R^{2s}}{2\Gamma(s)} [\zeta_R(-1+i+2s) + \zeta_R(-2+i+2s)] \\ &\quad \times \sum_{a=0}^{2i} x_{i,a} \frac{(i+a)\Gamma\left(s + \frac{i+a}{2}\right)}{\Gamma\left(1 + \frac{i+a}{2}\right)}. \end{aligned} \quad (2.18)$$

At this point, in order to find the heat-kernel coefficients of the Dirac operator on the ball with local boundary conditions, one has to use furthermore only

$$\begin{aligned} \zeta_R(1+\epsilon) &= \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0), \\ \Gamma(\epsilon-n) &= \frac{1}{\epsilon} \frac{(-1)^n}{n!} + \mathcal{O}(\epsilon^0). \end{aligned}$$

The final results for the relevant residues respectively function values are listed below for $k \in \mathbb{N}$. For $A_{-1}^{1/2}(s)$ and $A_0^{1/2}(s)$ we have

$$\begin{aligned}\text{Res } A_{-1}^{1/2} \left(\frac{3}{2} - k \right) &= \frac{(-1)^{k-1}}{(k-1)!} \frac{R^{3-2k}}{2\sqrt{\pi}\Gamma\left(\frac{5}{2} - k\right)} \zeta_R(1-2k), \\ \text{Res } A_0^{1/2} \left(\frac{3}{2} - k \right) &= \frac{(-1)^{k-1}}{(k-2)!} \frac{R^{3-2k}}{2\sqrt{\pi}\Gamma\left(\frac{5}{2} - k\right)} \zeta_R(1-2k), \quad (\text{for } k > 2).\end{aligned}$$

For the even indices i and for $n = 1, \dots, k-2$ it reads

$$\begin{aligned}\text{Res } A_{2n}^{1/2} \left(\frac{3}{2} - k \right) &= -\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2} - k\right)} \zeta_R(1+2n-2k) \\ &\quad \times \sum_{a=0}^{s(k)} x_{2n,2a+1} \frac{(-1)^{k-a-n}(2n+2a+1)}{(k-2-a-n)! \Gamma\left(\frac{3}{2} + n + a\right)},\end{aligned}$$

where the summation index $s(k)$ for $k \geq 3n+1$ is given by $s(k) = 2n-1$, whereas for $k < 3n+1$ one has $s(k) = k-2-n$, and

$$\text{Res } A_{2k}^{1/2} \left(\frac{3}{2} - k \right) = -\frac{R^{3-2k}}{4\Gamma\left(\frac{3}{2} - k\right)} \sum_{a=0}^{4k} x_{2k,a} \frac{(2k+a)\Gamma\left(\frac{3+a}{2}\right)}{\Gamma\left(\frac{a}{2} + 1 + k\right)}.$$

Finally the contributions for odd indices, $n = 1, \dots, k-1$, read

$$\begin{aligned}\text{Res } A_{2n-1}^{1/2} \left(\frac{3}{2} - k \right) &= -\frac{R^{3-2k}}{2\Gamma\left(\frac{3}{2} - k\right)} \zeta_R(1+2n-2k) \\ &\quad \times \sum_{a=0}^{\bar{s}(k)} x_{2n-1,2a} \frac{(-1)^{k-1-a-n}(2n+2a-1)}{(k-1-a-n)! \Gamma\left(\frac{1}{2} + n + a\right)},\end{aligned}$$

with $\bar{s}(k) = 2n-1$ for $k \geq 3n$ and $\bar{s}(k) = k-n-1$ for $k < 3n$, and

$$\text{Res } A_{2k-1}^{1/2} \left(\frac{3}{2} - k \right) = -\frac{R^{3-2k}}{4\Gamma\left(\frac{3}{2} - k\right)} \sum_{a=0}^{4k-2} x_{2k-1,a} \frac{(2k-1+a)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(\frac{1+a}{2} + k\right)}.$$

For the function values we found $A_{-1}^{1/2}(0) = -1/120$, $A_0^{1/2}(0) = 1/24$, $A_{-1}^{1/2}(-l) = A_0^{1/2}(-l) = 0$ for $l \in \mathbb{N}$, furthermore for $n = 1, \dots, k-1$, and $k \in \mathbb{N}$,

$$A_{2n}^{1/2}(1-k) = \frac{R^{2-2k}}{(k-1)!} \zeta_R(1+2n-2k) \sum_{a=0}^{s(k)} x_{2n,2a} \frac{(-1)^{n+a+1}}{(k-n-a-1)!(a+n-1)!},$$

with $s(k) = 2n$ for $k \geq 3n+1$, respectively $s(k) = k-n-1$ for $k < 3n+1$, and

$$\begin{aligned}A_{2n-1}^{1/2}(1-k) &= \frac{R^{2-2k}}{(k-1)!} \zeta_R(-1+2n-2k) \\ &\quad \times \sum_{a=0}^{\bar{s}(k)} x_{2n-1,2a+1} \frac{(-1)^{n+a+1}}{(k-n-a-1)!(a+n-1)!},\end{aligned}$$

where $\bar{s}(k) = 2n - 2$ if $k \geq 3n - 1$ and $\bar{s}(k) = k - n - 1$ if $k < 3n - 1$. Finally

$$A_{2k}^{1/2}(1-k) = \frac{(-1)^k R^{2-2k}}{4(k-1)!} \sum_{a=0}^{4k} x_{2k,a} \frac{(2k+a)\Gamma(1+\frac{a}{2})}{\Gamma(1+k+\frac{a}{2})},$$

and

$$A_{2k+1}^{1/2}(1-k) = \frac{(-1)^k R^{2-2k}}{4(k-1)!} \sum_{a=0}^{4k+2} x_{2k+1,a} \frac{(2k+a+1)\Gamma(\frac{3+a}{2})}{\Gamma(k+\frac{3+a}{2})}.$$

A list of the first coefficients is given in the appendix A. Especially the result given in [21], $\zeta_{1/2}^{lo}(0) = 11/360$, is confirmed.

For the spin- $\frac{1}{2}$ field with spectral boundary conditions as for the spin-1, spin- $\frac{3}{2}$, and spin-2 field exactly the same method may be employed (for the conditions analog to (2.1) see [27, 28]). Thus there is no need to present details of the calculation. Let us only mention, that the $\zeta(0)$ values agree with the values published already before (for a summary see [21]). Some of the heat-kernel coefficients corresponding to the values of the zeta function on the negative axis are summarized in the appendix A where we also give all necessary elements to perform the computation. An arbitrary number of coefficients may be obtained without any problem.

In [16] the sum rule

$$\zeta_{3/2}^{sp}(0) - \zeta_{1/2}^{sp}(0) = 2[\zeta_1(0) - 2\zeta_0(0)], \quad (2.19)$$

valid also for all coefficients in the heat-kernel expansion, has been stated. This might be checked for some coefficients using the results of the present work and of [11]. Actually, as may be seen in appendix A, for odd indices Eq. (2.19) represents two sum rules, namely one for the terms proportional to $\sqrt{\pi}$ and one for the terms proportional to $1/\sqrt{\pi}$. The term $1/\sqrt{\pi}$ is absent for the scalar case and so these parts cancel separately for the other spins.

3 Functional determinants for higher-spin fields on the 4-ball

Let us also concentrate in this section on the spin- $\frac{1}{2}$ field with local boundary conditions. For higher-spins we only write down the results at the end of the section.

In order to calculate the functional determinant of the relevant operator (Laplace, Dirac, ...), Eq. (2.10) is a very suitable starting point. Using the definition (1.2) we have to choose $N = 3$ in Eq. (2.11) in order to obtain an analytic representation of $\zeta_{\frac{1}{2}}^{lo}(s)$ around $s = 0$. Explicitly the relevant polynomials read

$$\begin{aligned} D_1(t) &= -\frac{1}{4}t + \frac{1}{12}t^3, \\ D_2(t) &= \frac{1}{8}t^3 + \frac{1}{8}t^4 - \frac{1}{8}t^5 - \frac{1}{8}t^6, \\ D_3(t) &= \frac{5}{192}t^3 + \frac{1}{8}t^4 + \frac{9}{320}t^5 - \frac{1}{2}t^6 \\ &\quad - \frac{23}{64}t^7 + \frac{3}{8}t^8 + \frac{179}{576}t^9. \end{aligned}$$

Using these, the contribution of the asymptotic terms to the functional determinant is found to be

$$\begin{aligned} \frac{d}{ds} \sum_{i=-1}^3 A_i^{1/2}(s) |_{s=0} &= -\frac{1597}{15120} - \frac{1}{180}\gamma - \frac{1}{1080}\pi^2 \\ &\quad - \frac{11}{180} \ln 2 + \frac{11}{180} \ln R - 2\zeta'_R(-3) - 3\zeta'_R(-2) - \zeta'_R(-1). \end{aligned}$$

For the part coming from $Z_{1/2}^{lo}(s)$ some additional calculation is needed. First one finds

$$\begin{aligned} Z_{1/2}^{lo,\nu'}(0) &= - \left[\ln \left(I_\nu^2(z\nu) + I_{\nu+1}^2(z\nu) \right) - 2\nu\eta - \sum_{n=1}^3 \frac{D_n(t)}{\nu^n} \right. \\ &\quad \left. - \ln \left(\frac{(1+z^2)^{1/2}}{\pi\nu z^2} (1-t) \right) \right] \Big|_{z=\frac{mR}{\nu}}. \end{aligned}$$

In the limit $m \rightarrow 0$ this gives

$$Z_{1/2}^{lo,\nu'}(0) = 2 \ln \Gamma(\nu+1) + 2\nu - 2\nu \ln \nu - \ln(2\pi\nu) + \sum_{n=1}^3 \frac{D_n(1)}{\nu^n}.$$

Use of the integral representation of $\ln \Gamma(\nu+1)$ [30], this may be rewritten as

$$Z_{1/2}^{lo,\nu'}(0) = 2 \int_0^\infty dt \left(-\frac{t}{12} + \frac{t^3}{720} + \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-t\nu}}{t}. \quad (3.1)$$

The remaining sum may be done using the techniques developed in connection with the scalar field. As an intermediate result one obtains

$$\begin{aligned} Z_{1/2}^{lo'}(0) &= 2 \lim_{z \rightarrow 0} \left[\frac{1}{360} \Gamma(z+1) \zeta_R(z+1) + 3\Gamma(z-2) \zeta_R(z-2) - 6\Gamma(z-3) \zeta_R(z-3) \right. \\ &\quad \left. + \frac{1}{360} \Gamma(z+2) \zeta_R(z+2) - \frac{1}{2} \Gamma(z-1) \zeta_R(z-1) \right. \\ &\quad \left. + \int_0^\infty dt \, t^z \frac{1}{e^t - 1} \left(\frac{d}{dt} + \frac{d^2}{dt^2} \right) \frac{1}{t(e^t - 1)} \right]. \end{aligned}$$

This gives

$$Z_{1/2}^{lo'}(0) = \frac{11}{90} + \frac{1}{180}\gamma + \frac{1}{1080}\pi^2 + \frac{8}{3}\zeta'_R(-3) + 3\zeta'_R(-2) + \frac{1}{3}\zeta'_R(-1).$$

Summing up, we have

$$\zeta_{1/2}^{lo'}(0) = \frac{251}{15120} - \frac{11}{180} \ln 2 + \frac{11}{180} \ln R + \frac{2}{3}\zeta'_R(-3) - \frac{2}{3}\zeta'_R(-1).$$

This result agrees with that of Apps, reported in [15] and found using a conformal transformation from the 4-hemisphere.

We retained the dependence on the radius R , because the coefficient of the $\ln R^2$ -term is known to be $\zeta(0)$ and this serves as a small check of the calculation

Once more all other cases may be treated in exactly the same way, so we list only the final results which are (we use the suffix ^{sp} to distinguish spectral boundary conditions from the local ones)

$$\zeta_{1/2}^{sp \prime}(0) = -\frac{2489}{30240} + \frac{1}{45} \ln 2 + \frac{11}{180} \ln R + \frac{2}{3}(\zeta_R'(-3) - \zeta_R'(-1)),$$

$$\zeta_1'(0) = -\frac{6127}{15120} - \frac{29}{45} \ln 2 - \frac{77}{90} \ln R - \ln \pi + \frac{2}{3}\zeta_R'(-3) - \zeta_R'(-2) - \frac{5}{3}\zeta_R'(-1),$$

$$\zeta_{3/2}^{lo \prime}(0) = \frac{2771}{15120} - \frac{71}{180} \ln 2 - 2 \ln \pi - \frac{289}{180} \ln R + \frac{2}{3}\zeta_R'(-3) - \frac{14}{3}\zeta_R'(-1),$$

$$\zeta_{3/2}^{sp \prime}(0) = -\frac{27689}{30240} - \frac{59}{45} \ln 2 - \frac{289}{180} \ln R - 2 \ln \pi + \frac{2}{3}\zeta_R'(-3) - \frac{14}{3}\zeta_R'(-1),$$

$$\zeta_2'(0) = -\frac{25027}{15120} + \frac{16}{45} \ln 2 - \frac{556}{45} \ln R - 7 \ln \pi + \frac{2}{3}\zeta_R'(-3) - \zeta_R'(-2) - \frac{23}{3}\zeta_R'(-1).$$

We would like to mention, that the contribution coming from $Z'_{1/2}(0)$ is identical for spectral and local boundary conditions. The difference is coming only from $A_0(s)$ to $A_3(s)$.

This concludes the list of our results for functional determinants of higher spin fields on the 4-dimensional Euclidean ball.

All results found agree with the recent results of Dowker [16].

4 Conclusions

In this article we applied the approach developed in [11, 12] to the case of higher-spin fields on the ball. We have seen, that the method is very well suited also for these cases and that no additional complication compared with the scalar field appears. We found that the $\zeta(0)$ value for the fermionic fields agree for local and spectral boundary conditions, however, other properties of the zeta function contained for example in the heat-kernel expansion and the zeta functional determinant are seen to differ for the two boundary conditions. Arbitrary spin-fields and also higher dimensional balls may be treated along the same lines.

Acknowledgments

We wish to thank Stuart Dowker for helpfully providing his results. KK thanks the Department of Theoretical Physics of the University of Trento for their kind hospitality at the final stage of this work.

A Heat-kernel coefficients for higher-spin fields

In this appendix we present the results for the spin-1/2, 1, 3/2 and the spin-2 fields obeying the indicated boundary conditions.

The λ eigenvalues of the operator (Laplace, Dirac,...) on the 4-dimensional ball for all the fields we are considering are the solutions of an equation $F(J_\nu, J_{\nu+1}) = 0$, which involves the Bessel functions. More precisely, depending on the boundary conditions, one has

$$J_\nu(\lambda) = 0 \quad (\text{A.1})$$

or

$$J_\nu^2(\lambda) - J_{\nu+1}^2(\lambda) = 0 \quad (\text{A.2})$$

where $\nu = n + a$ is positive and $n \geq 0$ (for convenience here we put $R = 1$). The degeneration of the eigenvalues is a polynomial of second degree in ν , that is $d_\nu = \alpha_0 + \alpha_1\nu + \alpha_2\nu^2$. The zeta function for the massless case reads

$$\zeta(s) = \sum_{n=0}^{\infty} d_\nu \int_{\gamma} \frac{dk}{2\pi i} k^{-2s} \frac{\partial}{\partial k} \ln F(J_\nu(k), J_{\nu+1}(k)) , \quad (\text{A.3})$$

and using the method explained in section 2, one finally gets

$$\begin{aligned} \zeta(s)\Gamma(s) &= \sum_k \frac{\alpha_0 \zeta_R(2s + k - 1, a) + \alpha_1 \zeta_R(2s + k - 2, a) + \alpha_2 \zeta_R(2s + k - 3, a)}{\Gamma(1 - s)} \\ &\quad \times \int_0^1 z^{-2s}(t) C_k(t) dt + \dots , \end{aligned} \quad (\text{A.4})$$

where $z(t)$ has been defined in section 2, while

$$\frac{d}{dt} \left[\ln z^\beta - \frac{1}{\nu} \ln F(I_\nu(z\nu), I_{\nu+1}(z\nu)) \right] \sim \sum_k \frac{C_k(t)}{\nu^k}, \quad (\text{A.5})$$

β being 1 or 2 according to whether F is given by Eq. (A.1) or (A.2). The dots in Eq. (A.4) stand for the analytic part in the neighbourhood of $s = \frac{4-n}{2}$, where the function on the left-hand side has a pole with residue K_n . In principle, using Eq. (A.4), one can compute K_n up to any order by a simple computer program. We easily found

Spin 1/2 - Local Boundary Conditions

$$J_\nu^2(\lambda) - J_{\nu+1}^2(\lambda) = 0, \quad \nu = n + 1, \quad d_\nu = \nu + \nu^2.$$

$$\begin{aligned} K_4 &= \frac{11}{360} \\ K_5 &= \frac{93\sqrt{\pi}}{32768} \\ K_6 &= \frac{1}{308} \\ K_7 &= \frac{8987\sqrt{\pi}}{6291456} \\ K_8 &= \frac{64627}{29099070} \end{aligned}$$

Spin 1/2 - Spectral Boundary Conditions

$$J_\nu(\lambda) = 0, \quad \nu = n + 1, \quad d_\nu = 2\nu + 2\nu^2.$$

$$\begin{aligned} K_4 &= \frac{11}{360} \\ K_5 &= \frac{817}{20160\sqrt{\pi}} - \frac{35\sqrt{\pi}}{32768} \\ K_6 &= \frac{24341}{1153152} \\ K_7 &= \frac{115069}{2306304\sqrt{\pi}} - \frac{911\sqrt{\pi}}{1572864} \\ K_8 &= \frac{5294503}{133024320} \end{aligned}$$

Spin 1 (Maxwell) - Dirichlet Boundary Conditions

$$J_\nu(\lambda) = 0, \quad \nu = n + 2, \quad d_\nu = -2 + 2\nu^2.$$

$$\begin{aligned} K_4 &= -\frac{77}{180} \\ K_5 &= -\frac{1}{8\sqrt{\pi}} - \frac{291\sqrt{\pi}}{32768} \\ K_6 &= -\frac{50549}{720720} \\ K_7 &= -\frac{25}{192\sqrt{\pi}} - \frac{4463\sqrt{\pi}}{1572864} \\ K_8 &= -\frac{13099069}{124156032} \end{aligned}$$

Spin 3/2 - Local Boundary Conditions

$$J_\nu^2(\lambda) - J_{\nu+1}^2(\lambda) = 0, \quad \nu = n + 2, \quad d_\nu = -1 + \nu + \nu^2.$$

$$\begin{aligned} K_4 &= -\frac{289}{360} \\ K_5 &= \frac{1}{2\sqrt{\pi}} + \frac{605\sqrt{\pi}}{32768} \\ K_6 &= \frac{53}{2772} \\ K_7 &= -\frac{5}{48\sqrt{\pi}} + \frac{48155\sqrt{\pi}}{6291456} \\ K_8 &= -\frac{12016999}{232792560} \end{aligned}$$

Spin 3/2 - Spectral Boundary Conditions

$$J_\nu(\lambda) = 0, \quad \nu = n + 2, \quad d_\nu = -8 + 4\nu^2.$$

$$\begin{aligned} K_4 &= -\frac{289}{360} \\ K_5 &= -\frac{4223}{20160\sqrt{\pi}} - \frac{547\sqrt{\pi}}{32768} \\ K_6 &= -\frac{32011}{274560} \\ K_7 &= -\frac{485531}{2306304\sqrt{\pi}} - \frac{8015\sqrt{\pi}}{1572864} \\ K_8 &= -\frac{796491}{4702880} \end{aligned}$$

Spin 2 (Transverse Traceless Modes) - Dirichlet Boundary Conditions

$$J_\nu(\lambda) = 0, \quad d_\nu = -8 + 2\nu^2.$$

$$\begin{aligned} K_4 &= -\frac{278}{45} \\ K_5 &= \frac{7}{4\sqrt{\pi}} - \frac{1059\sqrt{\pi}}{32768} \\ K_6 &= \frac{305699}{360360} \\ K_7 &= \frac{139}{96\sqrt{\pi}} - \frac{15119\sqrt{\pi}}{1572864} \\ K_8 &= \frac{87568801}{103463360} \end{aligned}$$

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